Thinking Mathematically

Integrating
ARITHMETIC & ALGEBRA
In Elementary School

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CONTENTS

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CHAPTER 1:
DEVELOPING MATHEMATICAL THINKING

Learning mathematics involves learning ways of thinking. It involves learning powerful mathematical ideas rather than a collection of disconnected procedures for carrying out calculations. But it also entails learning how to generate those ideas, how to express them using words and symbols, and how to justify to oneself and to others that those ideas are true. Elementary school children are capable of learning how to engage in this type of mathematical thinking, but often they are not given the opportunity to do so. In the chapters that follow, we portray how elementary school students can be encouraged to make explicit the powerful, unifying ideas of mathematics. We illustrate how they construct representations of those ideas in order to examine them and to communicate them, and we consider how students struggle to justify that the ideas are true.

INTEGRATING ARITHMETIC AND ALGEBRA

Arithmetic is and will continue to be a major focus of the elementary school curriculum. But we need to reconsider how arithmetic is taught and learned. Often the learning of arithmetic is isolated from other related mathematical ideas. The artificial separation of arithmetic and algebra deprives students of powerful ways of thinking about mathematics in the early grades and' makes it more difficult for them to learn algebra in the later grades. Understanding takes a long time to develop. The kind of mathematical thinking that can provide a foundation for learning algebra must be developed over an extended period of time, starting in the early elementary grades. In the chapters that follow, we discuss how to help children build on their implicit knowledge of arithmetic to provide a foundation for learning algebra. This does not mean simply using the current high school algebra curriculum in elementary school. To the contrary, rather than teaching algebra procedures to elementary school children, our goal is to support them in developing ways of thinking about arithmetic that are more consistent with the ways that students have to think to learn algebra successfully. These ways of thinking both pave the way for learning algebra and enhance the learning of arithmetic.

> Many adults equate school algebra with symbol manipulation-solving complicated equations and simplifying algebraic expressions. Indeed, the algebraic symbols and the procedures for working with them are a towering, historic mathematical accomplishment and are critical in mathematical work. But algebra is more than moving symbols around. Students need to understand the concepts of algebra, the structures and principles that govern the manipulation of the symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations. (National Council of Teachers of Mathematics, 2000, p. 37)

If students genuinely understand arithmetic at a level at which they can explain and justify the properties they are using as they carry out calculations, they have learned some critical foundations of algebra. Unfortunately, the way that most students learn arithmetic does not provide a foundation for learning algebra. For many students, arithmetic is perceived as a series of calculations. Students do not think much about the properties of number that make the calculations possible. Consequently, when they begin to study algebra, they do not see that the procedures they are using to solve equations and simplify expressions are based on the same properties of number that they have used in arithmetic. Even worse, the conceptions of arithmetic that many students bring to algebra can actually get in the way of their learning algebra. It does not have to be that way. Children can learn arithmetic in a way that provides a basis for learning algebra.

The fundamental properties that children use in carrying out arithmetic calculations provide the basis for most of the symbolic manipulation in algebra. For example, when children use their knowledge of simple arithmetic combinations to add multiples of ten, they are implicitly using a basic property that relates addition and multiplication. For example, a student might explain: "Fifty plus 30 equals 80, because 5 plus 3 equals 8, so 5 tens plus 3 tens is 8 tens, or 80." Symbolically, this can be represented as:

$$50 + 30 = 5 \times 10 + 3 \times 10 = (5 + 3) \times 10 = 8 \times 10 = 80$$

The same basic reasoning applies to simplifying algebraic expressions: $5b + 3b = 8b$, because $5b + 3b$ means 5 times b added to 3 times b. Represented symbolically:

$$5b + 3b = (5 + 3)b = 8b$$

or equivalently $5 \times b + 3 \times b = (5 + 3) \times b = 8 \times b$

Thus, understanding addition of multidigit numbers can provide a foundation for students to understand the principles that they will use to simplify algebraic expressions.

Focusing on fundamental properties of number and number operations also makes the learning of arithmetic more efficient and provides students with more powerful and flexible ways to apply the arithmetic they learn. Students who are successful in mathematics are not simply better at computing or manipulating symbols. By making generalizations and recognizing relationships among concepts and procedures, these students have less to learn and frequently are able to simplify their calculations. For example, students who understand the relation between multiplication and division can draw on this knowledge to simplify the learning of multiplication and division number facts. They can use what they know about 9 x 7 to figure out 63/9. Similarly, students who recognize that they can reorder and regroup numbers when they add them can simplify the calculation 75 + 48 + 25 by first adding the 75 and the 25.

STRUCTURE OF THE BOOK

We start by considering what children already know about fundamental properties of arithmetic and how they may use this knowledge to solve problems. Initially students are given the opportunity to use basic properties of arithmetic without explicitly identifying the properties they are using. Next we consider how to engage students in articulating conjectures about properties that they think are true and provide them opportunity and the means to express these conjectures clearly and accurately using words and symbols. In the concluding chapters, we examine how students learn to justify that the conjectures they have proposed are true and how to use them to show that the computational (and ultimately algebraic) procedures they use are valid.

What Children Know

We begin by examining some of the conceptions and misconceptions that students bring to learning mathematics throughout the elementary grades. Most students have some informal knowledge about basic properties of arithmetic, and we can build on this knowledge as we try to help students articulate fundamental ideas of arithmetic. Students also bring some conceptions about mathematics and the use of symbols that are not entirely accurate and need to be addressed so that they do not get in the way of students' learning. In Chapter 2, we consider how to address a common misconception about the meaning of the equal sign (see Box 1.1). In Chapter 3, we look at how students can use basic properties of arithmetic to simplify calculations. We initially focus on helping children look for relations among numbers and number operations using basic properties of arithmetic (see Box 1.1). The emphasis is on providing opportunities for children to begin to think about the properties of operations and relations rather than calculating by rote. At this point, the discussion is relatively informal, and children are not asked to be explicit about the number properties they are using to simplify calculations arid express relations.

BOX 1.1 *Thinking About Relations and Equality*

Consider how one second grade student, Robin, solved the following problem:

$18 + 27 = \square + 29$

Robin: Twenty-nine is two more than 27, so the number in the box has to be two less than 18 to make the two sides equal. So it's 16.

Robin's response is notable in several ways. First, Robin recognized that the equal sign represented a relation between the two expressions on either side of the equal sign. Many children throughout the elementary and middle school grades, and even into high school, think that the equal sign should be followed by the answer to the calculation on the left side of the equal sign. For the above problem, they would respond that 45 should go in the box. Although this error is common, it represents a serious misconception that is incompatible with the way that the equal sign is used in algebra and it limits students' learning of arithmetic. One of the keys to developing mathematical thinking is to understand that the equal sign expresses a relation.

Another notable feature of Robin's response is that she did not actually carry out the indicated calculations. She could have responded correctly by computing the sum of 18 and 27 and then figuring out what number to add to 29 to get 45. But she recognized that it was not necessary to do the calculation. Instead she compared the numbers in the expression and realized that because 29 was two more than 27, the number added to it had to be two less than 18 to make both expressions equal. Rather than carrying out the indicated calculations, she looked for relations between the expressions that could simplify the calculation. For Robin, the expression 18 + 27 did not just represent an arithmetic procedure to be carried out, the expression itself was an object of reflection that could be compared directly to the expression $\square + 29$. This type of thinking is critical in algebra and it can enrich the learning of arithmetic and allow students to be more flexible in applying their computational skills.

Making Generalizations Explicit and Expressing Them Using Words and Symbols

The procedures we use to add, subtract, multiply, divide, and compare numbers are based on a small number of fundamental properties of number and number operations, and much of algebra is based on the same basic properties. When students clearly understand these properties and how they apply to the mathematics they learn, they have acquired the basis for understanding arithmetic and algebra. As children learn arithmetic, they implicitly use a number of these fundamental properties. For example, when young children add a larger number to a smaller number, they often count on from the larger number even though the smaller number comes first. To add 3 + 8, they count on "8, [pause] 9, 10, 11." By counting on from the larger number, children essentially change the order of the numbers they add. This example suggests that, as early as first grade, many children have some implicit knowledge of an important property of addition, and they use this property and a number of other properties in carrying out their calculations.

Our goal is to make these properties the explicit focus of attention so that

- all students have access to basic mathematical properties;
- students understand why the computation procedures they use work
- the way they do;
- students apply their procedures flexibly in a variety of contexts; and
- students recognize the connections between arithmetic and algebra and can use their understanding of arithmetic as a foundation for learning algebra with understanding.

Instructional programs from prekindergarten through grade 12 should enable all students to –

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking of others;
- use the language of mathematics to express mathematical ideas precisely.

(NCTM, 2000, p. 60)

COMMUNICATION STANDARDS

We not only want to make explicit the fundamental properties of arithmetic; we want students to learn the importance of expressing them precisely and accurately using words and symbols. In Chapter 4, we discuss how students can be encouraged to articulate explicitly conjectures about basic number properties and how they may engage in discussion to refine the wording of their conjectures. Symbols provide a concise and precise means of representing basic properties about number operations and relations. In Chapters 5 and 6, we discuss how students can be introduced to symbolic notation that they can use for this purpose. The symbols of algebra play a dual role in students' learning. Students use symbols to represent basic properties of arithmetic, and students use these properties to simplify symbolic expressions and solve equations.

Justifying Mathematical Statements and Procedures

One of the key components of understanding is being able to explain why a procedure works or why a particular statement is true. There are standards in mathematics for deciding whether a mathematical statement is true or not. An important goal of mathematics instruction is to help students address how disputes in mathematics are resolved and understand what is required to show that a mathematical statement is true. This is the goal of Chapters 7, 8, and 9.

ENGAGING IN THE PRACTICES OF MATHEMATICS

We have found that students who learn to articulate and justify their own mathematical ideas, reason through their own and others' mathematical explanations, and provide a rationale for their answers develop a deep understanding that is critical to their future success in mathematics and related fields. Students not only need to learn the big ideas of mathematics; they need to learn the mathematical ways of thinking that are entailed in generating these ideas, in deciding how to express them, in justifying that they are true, and in using them to justify the mathematical procedures they are learning. They can learn these ways of thinking only by engaging in them.

REASONING AND PROOF STANDARD

Instructional programs from prekindergarten through grade 12 should enable all students to –

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proofs.

(NCTM, 2000, p. 56)

Algebra is sometimes characterized as generalized arithmetic. We are proposing that the teaching and learning of arithmetic be conceived as the foundation for algebra. Our goal is to develop ways of thinking about mathematics and of engaging in discussion about mathematics that are more productive both for learning arithmetic and for smoothing the transition to learning algebra. Algebra often serves as a gatekeeper that prevents students from continuing the study of mathematics, thereby limiting their access to college majors and careers that require knowledge of mathematics beyond simple arithmetic. Developing mathematical thinking in the elementary grades puts students on a path to learning mathematics with understanding so that algebra is a gateway to opportunity, not a gate that blocks their way.

REFERENCES

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics

TEACHER COMMENTARY 1.1

Mathematics is in everyone's life and should be a part of everyone's life. Unfortunately, there is a gate to higher level mathematics and it keeps certain kids out. Not enough kids are taking higher level math courses and it tends to be our disadvantaged kids who are losing out. This gate must be opened so that higher level mathematics is available and reachable for everyone. One way to open this gate is by talking about big ideas in the early grades instead of waiting until the upper grades. By middle and high school we have turned many kids off mathematics. Kids have already decided that either they don't like math anymore, or they don't understand math or they can't do math. The kids who have always been the mathematicians of the world have known implicitly the big ideas; it is just a part of them. There has also been a group of kids who had inklings of the big ideas but didn't develop their understanding of them. And then we have a group of kids who maybe didn't thing about big ideas. These kids might have even realized that there were big ideas in mathematics. One thing that is so helpful to all the children I teach is that we have a lot of conversations so that the big ideas are made explicit. The kids who have these ideas are putting them out there and, as they do so, clarifying their ideas. When the ideas get expressed, the kids who had the inklings are making relational bridges to the ideas they have and these ideas continue to develop. The kids who have never thought about these big ideas start to think about them; a seed has been placed. The more you talk about these ideas, the more the mathematics makes sense to them. I think there is a growth in all kids when you talk about the big ideas.

Annie Keith, Second and third grade teacher

CHAPTER 2:
EQUALITY

How would the students in your class respond to the following question:

What number would you put in the box to make this a true number sentence?

8 + 4 = □ + 5

If they are like most elementary students, they will not immediately respond that the answer is 7. Two common responses likely will be 12 and 17. Results from students in thirty typical elementary-grade classes are summarized in Table 2.1.

TABLE 2.1 ***Percent of Students Offering Various Solutions To 8 + 4 = □ + 5***

Response/Percent Responding[1]				
Grade	7	12	17	12 and 17
1 and 2	5	58	13	8
3 and 4	9	49	25	10
5 and 6	2	76	21	2

[1]***From Falkner, Levi, & Carpenter, 1999.***

Fewer than 10 percent of the students in any grade gave the correct response of 7, and strikingly, performance did not improve with age. In fact, in this sample, results for the sixth-grade students were actually slightly worse than the results for students in the earlier grades. These data illustrate that many, if not most, elementary school students harbor serious misconceptions about the meaning of the equal sign. Most elementary students; and many older students as well, do not understand that the equal sign denotes the relation between two equal quantities. Rather, they interpret the equal sign as a command to carry out a calculation, much as a calculator does when we press the equal sign. This misconception limits students' ability to learn basic arithmetic ideas with understanding and their flexibility in representing and using those ideas, and it creates even more serious problems as they move to algebra.

In the remainder of this chapter, we discuss students' conceptions of the meaning of the equal sign and consider briefly the potential sources of children's misconceptions. A primary objective of the chapter is to think about how we might engage children in examining and revising their conceptions of the meaning of the equal sign. We also examine in more depth what children's errors tell us about their understanding of basic mathematical principles. The good news is that we can make a significant difference in helping children interpret the equal sign in appropriate ways. Furthermore, discussions with students about their conceptions of what the equal sign means offer an ideal opportunity to begin to engage them in discussions that illustrate how mathematical ideas emerge and how mathematical disputes are resolved.

CHILDREN'S CONCEPTIONS OF THE MEANING OF THE EQUAL SIGN

Following are five typical responses to the problem given at the beginning of the chapter. The responses illustrate quite different conceptions of what the equal sign means.

Lucy: The Answer Comes Next

Ms. L.: Can you tell me what number you would put in the box to make this a true number sentence?

8 + 4 = □ + 5

Lucy: [After a brief period] Twelve.

Ms. L.: How do you know it is 12?

Lucy: Because that's the answer; 8 and 4 are 12. See, I counted, 8 [pause] 9, 10, 11, 12. See, that's 12.

Mrs. L.: What about this 5 over here? *[Pointing to the 5 in the number sentence]*

Lucy: That's just there.

Mrs. L.: Do you have to do anything with it?

Lucy: No. It's just there. It doesn't have anything to do with the 8 and 4.

Mrs. L.: What do you think it means?

Lucy: I don't know. I don't think it means anything. Maybe they just put it there to confuse us. You know, sometimes Ms. J. puts extra numbers in story problems to make us think about what to add or subtract.

For Lucy, the equal sign meant the answer came next, and the answer was the answer to the operation that came right before the equal sign. The equal sign was a command to carry out the calculation; it did not represent a relation between the numbers represented by 8 + 4 and □ + 5.

Randy: Use All the Numbers

Randy: It's 17.

Ms. L.: How did you figure 17?

Randy: Because I know that 8 and 4 is 12, and 5 more is 17.

Ms L.: Why, did you add all those numbers?

Randy: Because it says to add. See. *[Points to the two + symbols]*

Ms. L: Okay. But these two numbers are over here on this side of- the equal sign *[points at the 8 + 4]* and the 5 is over here *[points at the 5]*.

Randy: Yeah, but you have to add all the numbers. That's what it says to do.

Randy took into account all the numbers, but he did not recognize that where the symbols appeared in the number sentence made a difference. The only way he could think of to use all the numbers was to add them all together.

Barb: Extend the Problem

Barb: *[Puts 12 in the box and then writes = 17 after the 5 (8 + 4 = 12 + 5 = 17).]*

Ms. L: Can you tell me what you did?

Barb: Sure. I did 8 plus 4, and that5 12, and then I had to add the 5, so that made 17.

Ms. L.: Does 8 plus 4 equal 12 plus 5?

Barb: 8 and 4 is 12 and you add the 5 to get 17.

What Barb did made perfect sense to her. Like Lucy, she carried out the calculation before the equal sign and wrote the answer in the box. Like Lucy, she also believed that the number after the equal sign represented the calculation to the left. But like Randy, she then wanted to take into account the 5, so she added it to the 12 and put the answer at the end of the number sentence after a second equal sign. She was using the equal signs to show sequences of calculations. She certainly did not consider whether 8 + 4 was equal to 17. Even adults sometimes use this type of notation when more than two numbers are combined in different ways. This, however, is an incorrect use of the equal sign.

Making Sense

Lucy, Randy, and Barb all had different ways of making sense of the number sentence they were asked to solve. Their methods were not unreasonable attempts to deal with an unfamiliar problem. In virtually all of the number sentences that Lucy and Barb had encountered up to that time, the answer always came right after the equal sign. They generalized from their experience that one of the "rules" for writing number sentences was that the answer comes right after the equal sign. Randy made a different generalization. He overgeneralized a valid property of addition to presume that the order of the symbols in a number sentence is not important. In fact, Randy appeared to think that, for all practical purposes, the equal sign was irrelevant.

In all three cases, the errors were errors of syntax, errors in the children's interpretation of the rules of how the equal sign is used to express relations between two numbers. The children could figure out that 8 + 4 and 7 + 5 both are equal to 12. But they interpreted 8 + 4 as an operation to be carried out rather than as one way to represent 12, and for them the equal sign signified that the answer was to be calculated and written to the right of the equal sign.

TEACHER COMMENTARY 2.1

My students struggle a lot with the equal sign, but it is a good struggle. I want my first graders to understand that "equal" doesn't mean "the answer comes next." At the beginning of the year I gave my students 8 + 4 = □ + 7 and they all said 12. They completely ignored the + 7. So I asked them, "What about this plus 7?"

About half of them said, "Oh, plus 7 equals 19."

Now it is December and some kids are starting to let go of the idea that "equals" means "the answer comes next." They all understand equally in concerete terms. If I have 4 red blocks and 5 green blocks on one plate and only 6 yellow blocks on the other plate, they all can figure out how to add blocks to the second plate to make the number of blcoks on the two plates equal. Some kids might use a lot of trial and error to figure that out, but they can do it. It's not that they don't understand making things equal; it's using the equal sign. Right from the beginning of first grade I will write things like 12 = 6 + 6 or 3 = 3 so that they can see the different ways to use the equal sign.

Dawn Michaels, First-grade teacher

The fact that these children's misconceptions revolve around the appropriate use of symbols rather than failure to understand relations among quantities does not, however, mean that the misconceptions are trivial or are easily overcome. In fact, children may cling tenaciously to the conceptions they have formed about how the equal sign should be used, and simply explaining the correct use of the symbol is not sufficient to convince most children to abandon their prior conceptions and adopt the accepted use of the equal sign. Before we consider how to challenge children's misconceptions about the meaning of the equal sign, however, we should look at the responses to the same problem of two children who recognize that the equal sign represents a relation between two numbers.

Ricardo and Gina: a Relational View

Ricardo: *[After a short pause]* It's 7.

Ms. L.: How do you know that it's 7?

Ricardo: Well, 8 and 4 is 12. So I had to figure out what to go with 5 to make 12, and I figured out that had to be 7.

Ms. L.: So why did you want to figure out what to put with 5 to make 12?

Ricardo: Because I had 12 over here *[pointing to the left of the equal sign]*, so I had to have 12 over here *[pointing to the right side of the equal sign]*. And 5 and 7 is 12.

Ms. L.: Some kids have told me that 12 should go in the box. What do you think about that?

Ricardo: That's not right. Twelve and 5 – that would be 17, and that’s not equal to 12. These two sides have got to be the same.

Gina: [Very quickly] Seven.

Ms. L.: How do you know that it’s 7?

Gina: Well, I saw that the 5 over here *[pointing to the 5 in the number sentence]* was one more than the 4 over here *[pointing to the 4 in the number sentence]*, so the number in the box had to be one less than the 8. So it’s 7.

Ms. L.: That’s very interesting. Let’s try another one. How about:

57 + 86 = □ + 84

Gina: *[Almost immediately]* That’s easy. Fifty-nine.

Ms. L.: Wow. That was quick.

Gina: Yeah. It's just like the other one. It’s just two more, because the 84 is two less.

Both Ricardo and Gina viewed the equal sign as expressing a relation between numbers. The expressions on both sides of the equal sign had to represent the same number. Critically, they did not impose other arbitrary rules that number sentences had to follow a certain form. They were comfortable with expressions involving addition (or other combinations of operations) on either side of the equal sign. In contrast, many students believe that number sentences should follow a certain form with an operation on the left and the answer on the right. When they encounter different forms of number sentences, such as 8 + 4 = □ + 5, students like Lucy, Randy, and Barb are forced to adapt their rules to respond to an unfamiliar context, and their adaptations often involve calculating answers rather than looking for relations.

Ricardo calculated the sum on the left side of the equation and found a number to put in the box that when added to 5 would give the same number. Gina recognized both that she was looking for a relation between the two sides of the equation and that a relation among the numbers in the two expressions made it unnecessary for her to actually carry out the calculations. Although both Gina arid Ricardo demonstrated that they understood the appropriate use of the equal sign, Gina's strategy shows greater understanding and more flexibility than Ricardo's does. Gina considered the relation between the two addition expressions in the equation, not just the relation between the answers to the two calculations. She was able to consider 8 + 4 as more than a calculation to be carried out. This ability to reflect on relations among mathematical expressions such as 8 + 4 and □ + 5 is critical for students to think more generally about arithmetic and to extend their knowledge of arithmetic to algebra.

Without further probing, we cannot be sure that Ricardo could not solve the problem as Gina did. With small numbers children often will either know the answer or carry out the calculation even though they could compare the expressions without calculating. It generally is easier to observe whether children actually need to calculate when larger numbers are used.

DEVELOPING CHILDREN'S CONCEPTIONS OF EQUALITY

In order to help children develop an appropriate conception of equality; we have found that it is productive to put them in a position to challenge their existing conceptions. This can be accomplished by engaging them in discussions in which different conceptions of the equal sign emerge and must be resolved. In these discussions, which may occur either in a whole-class setting or in smaller groups, students can be encouraged to dearly articulate their conceptions of how the equal sign is used and to make them explicit so that the other members of the class can understand the different perspectives represented in the class. It is necessary for students to be clear about their conceptions of the equal sign and what supports those conceptions in order to attempt to figure out how to resolve the differences.

These discussions both address students' conceptions of the meaning of the equal sign and represent a first step in developing ways of thinking and conversing that embody the principles of algebraic reasoning. The contradictions between students' different conceptions of the meaning of the equal sign provide a context in which they need to examine and resolve inconsistencies. In discussing alternative conceptions of the equal sign, students are put in a position in which they need to articulate mathematical principles that often are left implicit. They must justify the principles that they propose in ways that convince others, and they must recognize and resolve conflicting assumptions and conclusions.

A Context for Discussing the Equal Sign

We have found that it is most fruitful to engage students in a discussion of the equal sign in the context of specific tasks. Appropriately chosen tasks can (a) provide a focus for students to articulate their ideas, (b) challenge students' conceptions by providing different contexts in which they need to examine the positions they have staked out, and (c) provide a window on children's thinking. True/false and open number sentences have proven particularly productive as a context for discussing equality. These number sentences can be manipulated in a variety of ways to create situations that may challenge students' conceptions and provide a context for discussion. At the beginning of this chapter, we saw how the open number sentence $8 + 4 = \square + 5$ generated a variety of responses that, in a classroom discussion, could provide a basis for considering alternative perspectives of the meaning of the equal sign. It is not easy, however, to resolve these different perspectives. The use of the equal sign is a matter of convention, and students cannot actually discover the convention or decide which perspective is right entirely by logic. However, a variety of contexts may encourage students to examine the positions they have taken.

True/False Number Sentences

Students may not have experience with true/false number sentences, but it is relatively easy to introduce them. We have found that it is not necessary to engage students in a general discussion about true/false number sentences or what it means for a number sentence to he true or false. It seems to work best to provide an example and ask whether the number sentence is true or false. It is a good idea to start with number sentences that involve relatively simple calculations with a single number to the right of the equal sign. Children quickly pick up the critical ideas in the context of talking about specific true/false number sentences, as in the following example.

Ms. B. Is this number sentence true or false: 8 - 5 = 3?

Students: True.

Ms. B.: How do you all know that it is true? Maria.

Maria: Because 8 minus 5 is 3.

Ms. B: Okay. How about this one, is this number sentence true or false: 3 x 4 = 15?

Chiang: No. That's false, because 3 x 4 is 12, and that's not equal to 15.

Ms. B.: Anybody disagree with Chiang? Okay, let's try another one, how about 587 + 468 = 1,055?

For this problem, the children had' to calculate the sum, but after a short time they concluded that this was a true number sentence because the sum they calculated was equal to 1,055.

Although these number sentences do not require students to supply the answer after the equal sign, they fit the form that students are familiar with in that there is a single number after the equal sign, and students generally do not have much difficulty deciding whether they are true or false. Once students are familiar with true/false number sentences, number sentences can be introduced that may encourage them to examine their conceptions of the meaning of the equal sign. Juxtaposing number sentences that students agree are true with those that are not in the familiar form may challenge some students' conceptions of how the equal sign is used. Following are several collections of problems that may bring out different perspectives on equality. Although the examples are shown with addition, number sentences with the same structure could be written for the other three operations as well.

a. 3 + 5 = 8
b. 8 = 3 + 5
c. 8 = 8
d. 3 + 5 = 3 + 5
e. 3 + 5 = 5 + 3
f. 3 + 5 = 4 + 4

Except for the first number sentence in the sequence, none of these examples follows the familiar form with an operation on two numbers on the left of the equal sign, which is followed directly by the answer. Asking students to choose whether each number sentence is true or false can encourage them to examine their assumptions about the equal sign. They generally agree that the numbers represented by the expressions on either side of the equal sign are the same, but they may still believe that these are not true number sentences; they often persist in asserting that you just cannot write it that way. These situations, however; often start to cause them some con1lict if they do not view the equal sign as expressing a relation meaning the same number as.

Most students will agree that the first sentence is true and that sentence provides a base of comparison for the other number sentences. They may not be so sure about the second, and they can be engaged in a discussion of how the two number sentences differ and why they do not believe the second is true. Comparing these sentences may help students to articulate the specific rules they are using to decide whether a sentence is true or false. They may, for example, say that there should just be a single number after the equal sign. In that case, Sentence *c* may encourage them to further examine their assumptions. If they respond that there have to be two numbers on the left of the equal sign, the number sentence 3 + 5 = 3 + 5 might provide a context for students to examine that assertion. Students often will agree that 3 + 5 = 3 + 5 is true because “there are the same numbers on both sides of the equal sign.” Sentence *e* provides a slightly different context for thinking about this claim.

Including zero in a number sentence may encourage students to accept a number sentence in which more than one number appears after the equal sign. For example, consider the following sequence of number sentences:

a. 9 + 5 = 14
b. 9 + 5 = 14 + 0
c. 9 + 5 = 0 + 14
d. 9 + 5 = 13 + 1

Once again most students will agree that the first sentence is true. Initially it may be easier for some students to accept Sentence *b* than Sentence *d*. For one thing, the “answer” to 9 + 5 appears right after the equal sign, and most students are familiar with the effect of adding zero. As a consequence, a number of students who would not accept Sentence *d* as true may accept Sentence *b* as true. This opens the door to having more than a single number after the equal sign. Sentence *u* offers an additional challenge. The “answer” no longer appears right after the equal sign, but the only difference between Sentences *b* and *c* is that the order of the numbers has been changed, and most students recognize that changing the order of numbers that they are adding does not change the answer. Thus, a comparison of Sentences *b* and *c* may provide an additional challenge to the rules that students impose on the use of the equal sign. If students are willing to accept Sentences *a, b,* and *c* as true, many of the reasons that they would give for not accepting Sentence *d* have already been eliminated. This can result in a potentially productive discussion about the necessity of applying rules consistently.

Open number sentences can be written to illustrate many of the same ideas. The question is: “What number can you put in the box to make this number sentence true?” Many numbers can be substituted for the box in each of the following number sentences, but only one makes the number sentence true. This way of talking about open number sentences provides a link among true/false and open number sentences. Following are open number sentences corresponding to the first collection of true/false number sentences:

a. 3 + 5 = □ or 3 + □ = 8
b. 8 = 3 + □ or □ = 3 + 5
c. 8 = □
d. 3 + 5 = □ + 5
e. 3 + 5 = □ + 3
f. 3 + 5 = □ + 4

These open number sentences, or similar problems with different numbers, might be given after the corresponding true/false question to follow up on the ideas that came out of the discussion of the true/false number sentence. We have found that students sometimes respond differently to true/false and open number sentences. Some students who have appeared to understand that the equal sign represents a relation when deciding whether a number sentence is true or false revert to calculating the answer to the left of the equal sign when they see number sentences like *d, e,* and *f* in the previous list.

Wording and Notation

We are trying to help students understand that the equal sign signifies a relation between two numbers. It sometimes is useful to use words that express that relation more directly: "Eight is the same amount as 3 plus 5." Some teachers also have found it useful to use notation that shows that the numbers on the two sides of the equal sign represent the same numerical value:

```
8 + 4 = 7 + 5        or      8 + 4 = 7 + 5
 \   /     \   /                   ||    ||
  12        12                   12 = 12
```

TEACHER COMMENTARY 2.2

Changing students' conceptions about the equal sign is not easy, as illustrated by this commentary written by a primary grade teacher.

My purpose for this lesson was to increase the children's comfort level with number sentences that had a single number on the left side of the equal sign and an expression involving two numbers and a plus or minus sign on the right side. Midway through the lesson I wrote the following number sentence and asked if it was true or false: 4 + 3 = 7. The children answered that it was true. Then I wrote this problem: 7 = 4 + 3. This time the children weren't nearly so sure, but eventually they decided that the sentence was true. At this point, a child suggested that I write 3 + 4 = 7. There was much discussion over whether that number sentence was the same or different from the first one I'd written. Generally the children did not have the language to discuss whether 4 + 3 was the same as 3 + 4 and why they thought it was or wasn't. I think the children simply considered that the since the number order was different, it was a different number sentence. In any event, all children agreed that his number sentence was true.

At this point the three number sentences were still on the board, each one under the other like this:

4 + 3 = 7

7 = 4 + 3

3 + 4 = 7

Since we had been talking about the equal sign, I considered this a golden opportunity to pose the following open number sentence: 3 + 4 = □ + 3. My question was, "What goes in the box?" Those children who were already comfortable with the meaning of the equal sign said 4, but the loudest response was 7!

Karen Falkner, First- and second-grade teacher

Classroom Interactions

Although the selection of tasks can provide a context for engaging students in examining their conceptions of the meaning of the equal sign, the nature of the discussion of mathematical ideas is critical. The goal is not just to teach students appropriate conceptions of the use of the equal sign; it is equally important to engage them in productive mathematical arguments.

Discussing alternative conceptions of the meaning of the equal sign can challenge students to examine the warrants that justify their claims. "Why do you think that?" "Why do you think that you cannot write number sentences that look like that?" Helping students understand the accepted use of the equal sign represents the first step in helping students make and justify generalizations about mathematics. The discussion of different conceptions of the equal sign and the ways that those differences are resolved can set the stage for beginning to help students understand the importance of carefully articulating and justifying mathematical claims. These discussions can provide opportunities for students to reflect upon and make sense of their ideas and to differentiate them from contrasting ideas of other students. They also can consider and evaluate the consequences of these ideas. We want children to see that there are different meanings held by different members of the class and that not all the meanings can be correct. The issue is not just to figure out what is correct but also to come to understand how fundamental differences get negotiated and resolved.

BENCHMARKS

Children do not necessarily pass through a sequence of distinct stages in developing their conceptions of the equal sign, and it should not be presumed that all children follow the same path to understanding how the equal sign is used. There are, however, some benchmarks to work toward as children's conception of the equal sign evolves.

1. Getting children to be specific about what they think the equal sign means represents a first step in changing their conceptions. In order for children to compare and contrast different conceptions, they need to be clear about what their conceptions are. This means getting beyond just comparing the different answers to a problem like $8 + 4 = \square + 5$. Some children may say that the equal sign must be preceded by two numbers joined by a plus or a minus and followed by the answer (resulting in an answer of 12 to this problem). Others may say that you have to use all the numbers (resulting in an answer of 17 to this problem). Even though neither of these conceptions is correct, getting them clearly articulated represents progress and provides a basis for challenging them.

2. The second benchmark is achieved when children first accept as true some number sentence that is not of the form $a + b = c$. It may be something like $8 = 5 + 3$, $8 = 8$, $3 + 5 = 8 + 0$, or $3 + 5 = 3 + 5$.

3. The third benchmark is achieved when children recognize that the equal sign represents a relation between two equal numbers. At this point they compare the two sides of the equal sign by carrying out the calculations on

each side of the equal sign. Ricardo's response is representative of this benchmark.

4. The fourth benchmark is achieved when children are able to compare the mathematical expressions without actually carrying out the calculations. Gina’s response is representative of this benchmark. The development of this type of reasoning is discussed in the next chapter.

These benchmarks are a guide. Not all children in any given class will attain these benchmarks at the same time, and not all children will follow this exact sequence. In fact, we have found that some children use relational thinking characteristic of level 4 as they are beginning to learn that the equal sign represents a relation between two numbers. Throughout the elementary grades, most classes initially will have a great deal of variability in the conceptions that students hold as they are striving to attain an understanding of the use of the equal sign.

WHAT TO AVOID

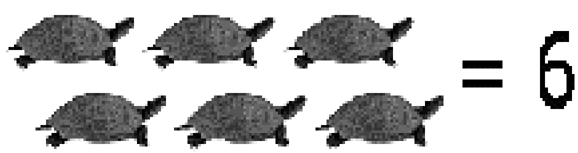
Frequently the equal sign is used as a shorthand for a variety of purposes. It is a good idea to avoid using the equal sign in ways that do not represent a relation between numbers. Table 2.2 includes several examples of uses of the equal sign that should be avoided.

TABLE 2.2 ***Inappropriate Uses of the Equal Sign***

1. **Listing the ages or some other numerical characteristic of people or things:**

   John = 8, Marcie = 9, etc.

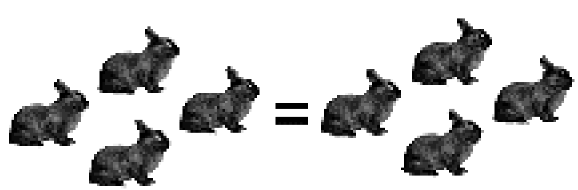
2. **Designating the number of objects in a collection:**



3. **Using equality to represent a string of calculations:**

   20 + 30 = 50 + 7 = 57 + 8 = 65

4. **Using equality between two pictures:**



In the first example, John is not a number. His age may be 8 or he may have scored 8 points in a game, but we should avoid using the equal sign to show this correspondence. The equal sign should be reserved to show the relation between numbers or expressions representing numbers. A similar concern applies to Examples 2 and 4. The number of objects in the collection is 6, but the set itself is not equal to the number 6. A group of six children is not equal to another group of six children. There is the same number of people in the two groups, but the groups themselves are not the same, and we should avoid using the equal sign in this way. It is tempting to use the equal sign to represent a series of calculations, as in the third example, but it is easy to see how that can reinforce some of the misconceptions illustrated at the beginning of this chapter. Rather than using the equal sign in this case, it is preferable to use write a longer version that emphasizes the correct use of the equal sign:

$$20 + 30 = 50$$

$$50 + 7 = 57$$

$$57 + 8 = 65$$

Once children begin to understand that the equal sign signifies a relation between numbers, it is important to continue to provide number sentences in a variety of forms and not to fall back to using only number sentences with the answer coming after the equal sign. Children's conceptions of how to use the equal sign tend to be fragile, and they continue to need experiences with number sentences that challenge them to think about the equal sign as signifying a relation rather than a signal to calculate an answer.

THE USE OF THE EQUAL SIGN IS A CONVENTION

In helping children to develop an understanding of the equal sign as expressing a relation, it is necessary to keep in mind that the way that the equal sign is used in mathematics is a matter of agreement and convention. Many properties of mathematics are not simply conventions. For example, the fact that the order of the numbers in an addition problem can be interchanged (the commutative property, e.g. $6 + 5 = 5 + 6$) is not simply a matter of convention, and in later chapters we will discuss how students can show that this property holds for all numbers. But it is not possible to justify that the equal sign represents a relation rather than a command to do something. If a student insists that you just cannot write number sentences that way, that it is not allowed, there is no way to refute that assertion directly. We can, however, ask the student to justify his or her claim that number sentences cannot be written that way. This can open the doors to a discussion of what should be taken as, evidence for a claim. Is it sufficient to presume that something is true because all the examples one has seen support that claim? Essentially, students are making claims about conventions, and the question is, how do they know that they have adopted the convention that is accepted by everyone else?

An important feature of conventions in mathematics is consistency. Once a convention has been agreed upon, it has to be applied consistently. Asking students to consider the implications of their conceptions of the equal sign for a variety of different forms of number sentences begins to put them in touch with this feature of mathematical thinking.

SMOOTHING THE TRANSITION TO ALGEBRA

A limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation. Consider, for example, the procedures for solving an equation like $5x + 32 = 97$. First one might subtract 32 from each side of the equation (or add -32). That requires understanding that the equal sign expresses a relation and that adding or subtracting the same thing to two expressions preserves the equality. When 32 is subtracted from both sides of the equation, the equation becomes $5x + 32 - 32 = 97 - 32$. What kind of meaning can students who exhibit the kinds of misconceptions of the equal sign illustrated at the beginning of this chapter attribute to this equation? With a limited conception of the equal sign as an instruction to carry out the preceding calculation, one is limited to carrying out arithmetic calculations. Understanding that the equal sign represents a relation between equal numbers opens up the power of algebra for representing problems and performing complex operations on mathematical expressions. This can enrich the learning of arithmetic as well as the learning of algebra.

WHERE DO CHILDREN'S MISCONCEPTIONS ABOUT THE EQUAL SIGN COME FROM?

It is difficult to sort out exactly why misconceptions about the meaning of the equal sign are so pervasive and so persistent. A good guess is that many children see only examples of number sentences with an operation to the left of the equal sign and the answer on the right, and they over generalize from those limited examples. But that is not the whole story. We have worked with kindergarten and first-grade children who have had limited exposure to number sentences of any kind but still strongly believed that the equal sign had to be followed immediately by the result of the calculation. Nevertheless, limiting children's exposure to a narrow range of number sentences does appear to contribute to students' misconceptions about the equal sign. In fact, we have found that even after students develop an appropriate conception of the meaning of the equal sign, they may revert to previous incorrect conceptions if they only see number sentences with the answer after the equal sign for an extended period of time.

Calculators may reinforce the notion that the equal sign means "carry out the preceding calculation" because they display an answer when the equal sign is pushed. It is doubtful, however, that children's misconceptions about the equal sign can be traced entirely to calculator use. Studies conducted before calculators were widely available found that children exhibited similar misconceptions about the equal sign.

A third possibility is that children may be predisposed to think of equality in terms of calculating answers rather than as a relation. Young children have an easier time carrying out a series of steps that lead to an answer than sorting out relations among quantities. However, the data summarized in Table 2.1 suggest that developing appropriate conceptions of the equal sign is not simply a matter of maturation. Unless we start to address directly children's concepts of equality; children's misconceptions will persist. On the other hand, we have found that children as young as first grade can learn the appropriate use of the equal sign. Furthermore, children's ability to understand the equal sign does not depend on their facility with computation. Children who are at relatively beginning stages in acquiring computational skills can still learn to use the equal sign appropriately. Developing an appropriate conception of the equal sign is something that we can and should work on for all children throughout the elementary grades.

CHILDREN'S CONCEPTIONS

The inappropriate generalizations about the equal sign that children make and often persist in defending are symptomatic of some fundamental limits in their understanding of how mathematical ideas are generated and justified. They have seen many examples of number sentences of a particular form, and they have overgeneralized from those examples. As they begin to justify conjectures about numbers and number relations, they tend to rely on examples and think that examples alone can prove their case. Thus, children's persistence in defending their conceptions of the meaning of the equal sign reflects their limited conceptions of how mathematical ideas are generated and justified.

It can be a challenge to get children to examine things that they believe to be true. Younger children, in particular, may have a difficult time considering several different ways of looking at equality and thinking about the consequences of the alternative perspectives. In general young children have difficulty in dealing with hypothetical situations and suspending judgment. But learning mathematics with understanding requires children to question why things work out the way that they do. As we shall see in the chapters to follow, children are more capable of reasoning and abstract thinking than they often have been given credit for. We need to appreciate the potential limits of children's thinking, but we do not want to impose limits on them because we do not anticipate what they are capable of. Time and again we have been surprised at what children can learn when we are attentive to what they are telling us.

CHALLENGES

1. What are the different responses that students may give to the following open number sentence: 9 + 7 = □ + 8?

2. Why did Ms. L. ask Gina to solve the open number sentence 57 + 86 = □ + 84 after she had solved the problem with smaller numbers? Are there any other questions you might have asked Gina or the other students described in this chapter to better understand their conceptions of the meaning of the equal sign and the relations between numerical expressions?

3. If you were the teacher in Teacher Commentary 2.2, what would you do next?

4. Design a sequence of true/false and/or open number sentences that you might use to engage your students in thinking about the equal sign. Describe why you selected the problems you did.

5. Discuss the problems you constructed for Challenge 4 with several students individually, a small group of students, or your class. Try to record or recall how specific students responded to the questions, what their responses suggested about their understanding of the meaning of the equal sign, and whether and how students' conceptions changed over the course of the

discussion. Were there any critical features of the lesson that seemed to have the most significant effect in changing children's conceptions of the meaning of the equal sign?

REFERENCES

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